OPERATOR-SPLITTING METHODS FOR WAVE EQUATIONS.

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Abstract.
The motivation for our studies is coming from the simulation of earthquakes, that are modeled by elastic wave equations. In our paper we focus on stiff phanomenons for the wave equations. In the course of this article we discuss iterative operator-splitting methods for wave equations motivated by realistic problems dealing with seismic sources and waves. The operator-splitting methods are well-known to solve this kind of multi-dimensional and multi-physical problems. We present the consistency analysis for iterative methods as theoretical background with respect to the underlying boundary conditions. From an algorithmic point of view we discuss the decoupling and non-decoupling method with respect to the eigenvalues. We verify our methods with test examples, for which analytical solutions can be derived. Multi-dimensional examples are presented for realistic applications for the wave equation. Finally we discuss the results.

Keywords: partial differential equations, operator-splitting methods, iterative methods, seismic sources and waves, consistency analysis.
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1. Introduction. Traditionally by using the classical operator-splitting methods we decouple the differential equation into more basic equations, in which each equation contains only one operator. These methods are often not sufficiently stable while also neglecting the physical correlations between the operators. We are going to develop new efficient methods based on a stable variant of iterative methods by coupling new operators and deriving new strong directions. We are going to examine the stability and consistency analysis for these methods and adopt them to linear acoustic wave equations (seismic waves).

The paper is organized as follows. A mathematical model based on the wave equation is introduced in Section 2. The utilized discretization methods are described in Section 3. A standard splitting method for the wave equation is given in Section 4. The splitting of the boundary conditions is discussed in Section 5. As a higher-order splitting method the LOD method is presented in Section 6 as well as the stability and consistency analysis for the spatial dependent case. We discuss the numerical results in Section 7. Finally we foresee our future works in the area of splitting and decomposition methods.

2. Mathematical model. The motivation for the study presented below is coming from a computational simulation of earthquakes, see [3], and the examination of seismic waves [1] and [2].

We concentrate on the scalar wave equation, see [11], for which the mathematical equations are given by

$$\partial_{tt}\, u \;=\; D_1(x,y)\,\partial_{xx}\, u \;+\; D_2(x,y)\,\partial_{yy}\, u \;+\; D_3(x,y)\,\partial_{zz}\, u \;\text{ in } \Omega, \tag{2.1}$$

$$u(x,y,0) = u_0(x,y)\;,\; u_t(x,y,0) = u_1(x,y) \;\text{ in } \Omega. \tag{2.2}$$

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The unknown function $u = u(x,t)$ is considered to be in $\Omega \times (0,T) \subset I\!R^d \times I\!R$, where the spatial dimension is given by d. The function $\mathbf{D}(x,y) = (D_1(x,y), D_2(x,y), D_3(x,y))^t \in I\!R^{3,+}$ describes the wave propagation in x, y, z. The functions $u_0(x,y)$ and $u_1(x,y)$ are the initial conditions for the wave equation.

We deal with the following boundary conditions:

$$u(x,y,t) = u_3, \text{ on } \partial\Omega \times T: \text{ Dirichlet boundary condition,} \tag{2.3}$$
$$\frac{\partial u(x,y,t)}{\partial n} = 0, \text{ on } \partial\Omega \times T: \text{ Neumann boundary condition,} \tag{2.4}$$
$$\mathbf{D}\nabla u(x,y,t) = u_{\mathrm{out}}, \text{ on } \partial\Omega \times T: \text{ outflow boundary condition.} \tag{2.5}$$

3. Discretization methods. At first we underly finite difference schemes for the time and space discretization.

For the classical wave equation this is the well-known discretization in time and space.

Based on this discretization, the time is discretized as follows:

$$U_{tt,i} = \frac{U_i^{n+1} - 2U_i^n + U_i^{n-1}}{\Delta t^2}, \tag{3.1}$$
$$U(0) = u_0, U_t(0) = u_1, \tag{3.2}$$

where the index i refers to the space point x_i and $\Delta t = t^{n+1} - t^n$ is the time step. The space is discretized with initial conditions as

$$U_{xx,n} = \frac{U_{i+1}^n - 2U_i^n + U_{i-1}^n}{\Delta x^2}, \tag{3.3}$$
$$U(0) = u_0, U_t(0) = u_1, \tag{3.4}$$

where the index n refers to the time t_n and $\Delta x = x_{i+1} - x_i$ is the grid width.

Then the two-dimensional equation,

$$u_{tt} = D_1 u_{xx} + D_2 u_{yy} \text{ in } \Omega, \tag{3.5}$$
$$u(x,y,0) = u_0(x,y), u_t(x,y,0) = u_1(x,y), \tag{3.6}$$
$$u(x,y,t) = u_2 \text{ on } \partial\Omega, \tag{3.7}$$

is discretized with the unconditionally stable implicit η-method, see [4]:

$$\begin{aligned} &\frac{U_{i,j}^{n+1} - 2U_{i,j}^n + U_{i,j}^{n-1}}{\Delta t^2} \\ &= \frac{D_1}{\Delta x^2}\left(\eta\left(U_{i+1,j}^{n+1} - 2U_{i,j}^{n+1} + U_{i-1,j}^{n+1}\right)\right. \\ &+(1-2\eta)\left(U_{i+1,j}^n - 2U_{i,j}^n + U_{i-1,j}^n\right) + \eta\left.\left(U_{i+1,j}^{n-1} - 2U_{i,j}^{n-1} + U_{i-1,j}^{n-1}\right)\right) \\ &+\frac{D_2}{\Delta y^2}\left(\eta\left(U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n\right)\right. \\ &+(1-2\eta)\left(U_{i,j+1}^n - 2U_{i,j}^n + U_{i,j-1}^n\right) + \eta\left.\left(U_{i,j+1}^{n-1} - 2U_{i,j}^{n-1} + U_{i,j-1}^{n-1}\right)\right), \end{aligned} \tag{3.8}$$

where Δx and Δy denote the grid width in x and y and $0 \leq \eta \leq 1$. The initial conditions are given by $U(x,y,t^n) = u_0(x,y)$ and $U(x,y,t^{n-1}) = u_0(x,y) - \Delta t u_1(x,y)$.

These discretization schemes are adopted to the operator-splitting schemes.

On the finite differences grid k corresponds to the time step, and h_x, h_y, h_z are the grid sizes in the different spatial directions. The time nk is denoted by t^n, and

i, j, l refer to the spatial coordinates of the grid point (ih_x, jh_y, kh_z). Let $\mathbf{u}^n$ denote the grid function on the time level n, and let $u^n_{i,j,l}$ be the specific value of $\mathbf{u}^n$ at point i, j, l. The value of the grid function during the iteration is denoted by an extra super script as $u^{n,m}_{i,j,l}$.

In the next section we describe the traditional splitting methods for the wave equation.

4. Traditional splitting methods. Our classical method is based on the splitting method of [4] and [9].

The classical splitting methods ADI (alternating direction methods) are based on the idea of computing the different directions of the given operators. Each direction is computed independently by solving more basic equations. The result combines all the solutions of the elementary equations. So we obtain more efficiency by decoupling the operators.

The classical splitting method for the wave equation is derived from

$$\begin{aligned} &\partial_{tt} u(t) = (A + B + C) u(t) + f(t), \\ &t \in (t^n, t^{n+1}), \quad u(t^n) = u_0, u'(t^n) = u_1, \end{aligned} \tag{4.1}$$

where the initial functions u_0 and u_1 are given. We can also apply for u_1 that $u'(t^n) = \frac{u(t^n) - u(t^{n-1})}{\Delta t} + \mathcal{O}(\Delta t) = u_1$. Consequently we have $u(t^{n-1}) \approx u_0 - \Delta t u_1$. The right-hand side $f(t)$ is given as a force term.

We could decouple the equation into 3 simpler equations obtaining a method of second order.

$$\begin{aligned} \tilde{\tilde{u}} - 2u(t^n) + u(t^{n-1}) &= \Delta t^2 A(\eta \tilde{\tilde{u}} + (1-2\eta) u(t^n) + \eta u(t^{n-1})) \\ &+ \Delta t^2 B u(t^n) + \Delta t^2 C u(t^n) \\ &+ \Delta t^2 (\eta f(t^{n+1}) + (1-2\eta) f(t^n) + \eta f(t^{n-1})), \end{aligned} \tag{4.2}$$

$$\begin{aligned} \tilde{u} - 2u(t^n) + u(t^{n-1}) &= \Delta t^2 A(\eta \tilde{\tilde{u}} + (1-2\eta) u(t^n) + \eta u(t^{n-1})) \\ &+ \Delta t^2 B(\eta \tilde{u} + (1-2\eta) u(t^n) + \eta u(t^{n-1})) + \Delta t^2 C u(t^n) \\ &+ \Delta t^2 (\eta f(t^{n+1}) + (1-2\eta) f(t^n) + \eta f(t^{n-1})), \end{aligned} \tag{4.3}$$

$$\begin{aligned} u(t^{n+1}) - 2u(t^n) + u(t^{n-1}) &= \Delta t^2 A(\eta \tilde{\tilde{u}} + (1-2\eta) u(t^n) + \eta u(t^{n-1})) \\ &+ \Delta t^2 B(\eta \tilde{u} + (1-2\eta) u(t^n) + \eta u(t^{n-1})) \\ &+ \Delta t^2 C(\eta u(t^{n+1}) + (1-2\eta) u(t^n) + \eta u(t^{n-1})) \\ &+ \Delta t^2 (\eta f(t^{n+1}) + (1-2\eta) f(t^n) + \eta f(t^{n-1})), \end{aligned} \tag{4.4}$$

where the result is given as $u(t^{n+1})$ with the initial conditions $u(t^n) = u_0$ and $u(t^{n-1}) = u_0 - \Delta t u_1$ and $\eta \in (0, 0.5)$. A fully coupled method is given for $\eta = 0$, for $0 < \eta \leq 0.5$ the decoupled method consists of a composition of explicit and implicit Euler methods.

The spatial discretization is given by

$$A = \frac{\partial^2}{\partial^2 x},\ B = \frac{\partial^2}{\partial^2 y},\ C = \frac{\partial^2}{\partial^2 z},$$

where the approximated discretization is given by the finite difference method as:

$$\begin{aligned} Au(x, y, z) &\approx \frac{u(x + \Delta x, y, z) - 2u(x, y, z) + u(x - \Delta x, y, z)}{\Delta x^2}, \\ Bu(x, y, z) &\approx \frac{u(x, y + \Delta y, z) - 2u(x, y, z) + u(x, y - \Delta y, z)}{\Delta y^2}, \\ Cu(x, y, z) &\approx \frac{u(x, y, z + \Delta z) - 2u(x, y, z) + u(x, y, z - \Delta z)}{\Delta z^2}. \end{aligned}$$

We have to compute the first equation 4.2 and get the result $\tilde{\tilde{u}}$, that is a further initial condition for the second equation 4.3, after whose computation we obtain $\tilde{u}$. In the third equation 4.4 we have to put $\tilde{c}$ as a further initial condition and get the result $u(t^{n+1})$.

The underlying idea consists of the approximation of pairwise operators:

$$\Delta t^2 A\eta(\tilde{\tilde{u}} - 2u(t^n) + u(t^{n-1})) \approx 0,$$
$$\Delta t^2 B\eta(\tilde{u} - 2u(t^n) + u(t^{n-1})) \approx 0,$$

which we can raise to second order.

5. Boundary splitting method. The time-dependent boundary conditons also have to be taken into account for the splitting method. Let us consider the three-operator example with the equations

$$\partial_{tt} u(t) = (A + B + C)u(t) + h(t),\ t \in (t^n, t^{n+1}), \tag{5.1}$$
$$u(t^n) = g(t), u'(t^n) = f(t), \tag{5.2}$$

where $A = D_1(x, y, z)\frac{\partial^2}{\partial x^2}$, $B = D_2(x, y, z)\frac{\partial^2}{\partial y^2}$ and $A = D_3(x, y, z)\frac{\partial^2}{\partial z^2}$ are the spatial operators. The wave propagation functions are given by $D_1(x, y, z), D_2(x, y, z)$, and $D_3(x, y, z) : I\!R^3 \rightarrow I\!R^+$.

Hence for three operators we have the following second-order splitting method:

$$\begin{aligned}
\tilde{\tilde{u}} - 2\tilde{\tilde{u}}(t^n) + \tilde{\tilde{u}}(t^{n-1}) =&\ \Delta t^2 A(\eta\tilde{\tilde{u}} + (1-2\eta)\tilde{\tilde{u}}(t^n) + \eta\tilde{\tilde{u}}(t^{n-1})) && (5.3)\\
&+\Delta t^2 B\tilde{\tilde{u}}(t^n) + \Delta t^2 C\tilde{\tilde{u}}(t^n)\\
&+\Delta t^2(\eta h(t^{n+1}) + (1-2\eta)h(t^n) + \eta h(t^{n-1})),\\
\tilde{u} - 2\tilde{u}(t^n) + \tilde{u}(t^{n-1}) =&\ \Delta t^2 A(\eta\tilde{\tilde{u}} + (1-2\eta)\tilde{u}(t^n) + \eta\tilde{u}(t^{n-1})) && (5.4)\\
&+\Delta t^2 B(\eta\tilde{u} + (1-2\eta)\tilde{u}(t^n) + \eta\tilde{u}(t^{n-1})) + \Delta t^2 C\tilde{u}(t^n)\\
&+\Delta t^2(\eta h(t^{n+1}) + (1-2\eta)h(t^n) + \eta h(t^{n-1})),\\
u(t^{n+1}) - 2\hat{u}(t^n) + \hat{u}(t^{n-1}) =&\ \Delta t^2 A(\eta\tilde{\tilde{u}} + (1-2\eta)\hat{u}(t^n) + \eta\hat{u}(t^{n-1})) && (5.5)\\
&+\Delta t^2 B(\eta\tilde{u} + (1-2\eta)\hat{u}(t^n) + \eta\hat{u}(t^{n-1}))\\
&+\Delta t^2 C(\eta u(t^{n+1}) + (1-2\eta)\hat{u}(t^n) + \eta\hat{u}(t^{n-1}))\\
&+\Delta t^2(\eta h(t^{n+1}) + (1-2\eta)h(t^n) + \eta h(t^{n-1})),
\end{aligned}$$

where the result is given as $u(t^{n+1})$.

The boundary values are given by

- Dirichlet values. We have to use the same boundary values for all three equations.
- Neumann values. We have to decouple the values into the different directions:

$$1)\frac{\partial\tilde{\tilde{u}}}{\partial n} = 0 \text{ is splitted in } \frac{\partial\tilde{\tilde{u}}}{\partial x}n_x + \frac{\partial\tilde{\tilde{u}}}{\partial y}n_y + \frac{\partial\tilde{\tilde{u}}}{\partial z}n_z = 0, \tag{5.6}$$

$$2)\frac{\partial\tilde{u}}{\partial n} = 0 \text{ is splitted in } \frac{\partial\tilde{\tilde{u}}}{\partial x}n_x + \frac{\partial\tilde{u}}{\partial y}n_y + \frac{\partial\tilde{u}}{\partial z}n_z = 0, \tag{5.7}$$

$$3)\frac{\partial u(t^{n+1})}{\partial n} = 0 \text{ is splitted in } \frac{\partial\tilde{\tilde{u}}}{\partial x}n_x + \frac{\partial\tilde{u}}{\partial y}n_y + \frac{\partial u^{n+1}}{\partial z}n_z = 0. \tag{5.8}$$

- Outflow values. We have to decouple the values into the different directions:

$$
\begin{aligned}
&1)\mathbf{nD}\nabla\tilde{\tilde{u}} = c_{out}, \\
&\quad\text{is splitted in } D_1\partial_x\tilde{\tilde{u}}n_x + D_2\partial_y\tilde{\tilde{u}}n_y + D_3\partial_z\tilde{\tilde{u}}n_z = u_{\text{out}}, && (5.9)\\
&2)\mathbf{nD}\nabla\tilde{u} = u_{\text{out}}, \\
&\quad\text{is splitted in } D_1\partial_x\tilde{\tilde{u}}n_x + D_2\partial_y\tilde{u}n_y + D_3\partial_z\tilde{u}n_z = u_{\text{out}}, && (5.10)\\
&3)\mathbf{nD}\nabla u^{n+1} = u_{\text{out}}, \\
&\quad\text{is splitted in } D_1\partial_x\tilde{\tilde{u}}n_x + D_2\partial_y\tilde{u}n_y + D_3\partial_z u^{n+1}n_z = u_{\text{out}}, && (5.11)
\end{aligned}
$$

where $\mathbf{n}$ is the outer normal vector and $\mathbf{D} = \begin{pmatrix} D_1 & 0 & 0 \\ 0 & D_2 & 0 \\ 0 & 0 & D_3 \end{pmatrix}$ is the parameter matrix to the wave propagations.

We have the following initial conditions for the three equations:

$$u(t^n) = u_0, \tag{5.12}$$

$$u(t^{n-1}) = u_0 - \Delta t u_1 + \frac{\Delta t^2}{2}((A+B)u_0) + \mathcal{O}(\Delta t^3), \tag{5.13}$$

$$u(t^{n-1}) = u_0 - \Delta t u_1 + \frac{\Delta t^2}{2}((A+B)(u_0 - \Delta t/3 u_1 + \frac{\Delta t^2}{12}(A+B)u_0)) + \mathcal{O}(\Delta t^5). \tag{5.14}$$

Remark 5.1. *By solving the two or three splitting steps it is important to mention, that each solution $\tilde{\tilde{u}}$, $\tilde{u}$ and u is corrected only once by using the boundary conditions.*

Otherwise an "overdoing" of the boundary conditions takes place.

6. LOD method: Locally one-dimensional method. In the follwoing we introduce the LOD method as an improved splitting method while using pre-stepping techniques.

The method was discussed in [11] and is given by:

$$u^{n+1,0} - 2u^n + u^{n-1} = \Delta t^2 (A+B)u^n, \tag{6.1}$$

$$u^{n+1,1} - u^{n+1,0} = \Delta t^2 \eta A(u^{n+1} - 2u^n + u^{n-1}), \tag{6.2}$$

$$u^{n+1} - u^{n+1,1} = \Delta t^2 \eta B(u^{n+1} - 2u^n + u^{n-1}), \tag{6.3}$$

where $\eta \in (0.0, 0.5)$ and A, B are the spatially discretized operators. The time step is equidistant and given as $\Delta t = t^{n+1} - t^n$.

If we eliminate the intermediate values in (6.1)- (6.3) we obtain

$$
\begin{aligned}
u^{n+1} - 2u^n + u^{n-1} &= \Delta t^2 (A+B)(\eta u^{n+1} - (1-2\eta)u^n + \eta u^{n-1}) \\
&\quad + B_\eta (u^{n+1} - 2u^n + u^{n-1}),
\end{aligned} \tag{6.4}
$$

where $B_\eta = \eta^2 \Delta t^2 (AB)$ is denoted as the local error of the splitting method and thus $B_\eta(u^{n+1} - 2u^n + u^{n-1}) = \mathcal{O}(\Delta t^4)$.

So we obtain a higher-order method.

Remark 6.1.

For $\eta \in (0.25, 0.5)$ we have unconditionally stable methods and to receive higher order we use $\eta = \frac{1}{12}$. Then for sufficiently small time steps we get a conditionally stable splitting method.

6.1. Stability and consistency analysis for the LOD method. The consistency of the fourth-order splitting method is given in the next theorem.

Hence we assume discretization orders of $\mathcal{O}(h^p)$, $p = 2, 4$, for the discretization in space, with $h = h_x = h_y$ being the spatial grid width.

Then we obtain the following consistency result for our method (6.1)-(6.3):

THEOREM 6.1. *The consistency of the LOD method is given by:*

$$u_{tt} - Au - (\overline{\partial_{tt}}u - \tilde{A}u) = \mathcal{O}(\Delta t^4), \tag{6.5}$$

where $\overline{\partial_{tt}}$ is a second-order discretization in time and $\tilde{A}$ is the discretized fourth-order spatial operator.

Proof. We add the equations (6.1)-(6.3) and obtain, see also [11]:

$$\overline{\partial_{tt}}u^n - \tilde{A}(\theta u^{n+1} + (1-2\theta)u^n + \theta u^{n-1}) - \tilde{B}(u^{n+1} - 2u^n + u^{n-1}) = 0, \tag{6.6}$$

where

$\tilde{B} = \theta^2 \Delta t^2 \tilde{A}_1 \tilde{A}_2$.

Therefore we obtain a splitting error of $\tilde{B}(u^{n+1} - 2u^n + u^{n-1})$.

Sufficient smoothness assumed we have $(u^{n+1} - 2u^n + u^{n-1}) = \mathcal{O}(\Delta t^2)$, and we obtain $\tilde{B}(u^{n+1} - 2u^n + u^{n-1}) = \mathcal{O}(\Delta t^4)$.

Thus we obtain a fourth-order method, if the spatial operators are also discretized as fourth-order terms.

□

The stability of the fourth-order splitting method is given in the following theorem.

THEOREM 6.2. *The stability of our method is given by:*

$$\begin{aligned} &||(1 - \Delta t^2 \tilde{B})^{1/2} \partial_t^+ u^n||^2 + \mathcal{P}^+(u^n, \theta) \\ \le\ &||(1 - \Delta t^2 \tilde{B})^{1/2} \partial_t^+ u^0||^2 + \mathcal{P}^+(u^0, \theta), \end{aligned} \tag{6.7}$$

where $\theta \in [0.25, 0.5]$ and

$\mathcal{P}^{\pm}(u^j, \theta) := \theta(\tilde{A}u^j, u^j) + \theta(\tilde{A}u^{j\pm 1}, u^{j\pm 1}) + (1-2\theta)(\tilde{A}u^j, u^{j\pm 1})$.

Proof.

We have to proof the theorem for a test function $\overline{\partial}_t u^n$, where $\overline{\partial}_t$ denotes the central difference.

For $n \ge 1$ we have

$$((1 - \Delta t^2 \tilde{B})\overline{\partial}_{tt}u^n, \overline{\partial}_t u^n) + (\tilde{A}(\theta u^{j+1} - (1-2\theta)u^j + \theta u^{j-1}), \overline{\partial}_t u^n) = 0. \tag{6.8}$$

Multiplying with Δt and summarizing over j yields:

$$\sum_{j=1}^{n} ((1 - \Delta t^2 \tilde{B})\overline{\partial}_{tt}u^j, \overline{\partial}_t u^j)\Delta t + (\tilde{A}(u^{j+1} - 2u^j + u^{j-1}), \overline{\partial}_t u^j)\Delta t = 0. \tag{6.9}$$

We can derive the identities,

$$
\begin{aligned}
&((1 - \Delta t^2 \tilde{B})\overline{\partial}_{tt} u^j, \overline{\partial}_t u^j)\Delta t \\
&= 1/2||(1 - \Delta t^2 \tilde{B})^{1/2} \partial_t^+ u^j||^2 - 1/2||(1 - \Delta t^2 \tilde{B})^{1/2} \partial_t^- u^h||^2,
\end{aligned} \tag{6.10}
$$

$$
\begin{aligned}
&(\tilde{A}(\theta u^{j+1} - (1 - 2\theta)u^j + \theta u^{j-1}), \overline{\partial}_t u^j)\Delta t \\
&= 1/2(\mathcal{P}^+(u^j, \theta) - \mathcal{P}^-(u^j, \theta)),
\end{aligned} \tag{6.11}
$$

and obtain the result

$$
\begin{aligned}
&||(1 - \Delta t^2 \tilde{B})^{1/2} \partial_t^+ u^n||^2 + \mathcal{P}^+(u^n, \theta) \\
&\leq ||(1 - \Delta t^2 \tilde{B})^{1/2} \partial_t^+ u^0||^2 + \mathcal{P}^+(u^0, \theta),
\end{aligned} \tag{6.12}
$$

see also the idea of [11].
□

REMARK 6.2. *For $\theta = \frac{1}{12}$ we obtain a fourth-order method.*

To compute the error of the local splitting we have to use the multiplier $\tilde{A}_1 \tilde{A}_2$, thus for large constants we have an unconditional small time step.

REMARK 6.3.
1. *The unconditinal stable version of the LOD method is given for $\theta \in [0.25, 0.5]$.*
2. *The truncation error is $\mathcal{O}(\Delta t^2 + h^p)$, $p \geq 2$ for $\theta \in [0, 0.5]$.*
3. *For $\theta = 1/12$ we have a fourth-order method in time $\mathcal{O}(\Delta t^2 + h^p), p \geq 2$.*
4. *For $\theta = 0$ we have a second-order explicit scheme.*
5. *The CFL-condition is important for all $\theta \in [0, 0.5]$ with*
   $CFL = \Delta t^2 / \Delta x_{max}^2 D_{max}$,
   where x_{max} is the maximal spatial step and D_{max} is the maximal wave-propagation parameter in space.

In the next section we apply our theoratical results to our model problems.

7. Numerical examples for the spatial splitting methods. The test examples are discussed with respect to analytical solutions, boundary conditions and spatially dependent propagation functions.

7.1. Test example 1: problem with analytical solution and Dirichlet boundary condition. We deal with a two-dimensional example with constant coefficients where we can derive an analytical solution.

$$
\partial_{tt} u = D_1^2 \partial_{xx} u + D_2^2 \partial_{yy} u, \tag{7.1}
$$

$$
u(x, y, 0) = u_0(x, y) = \sin(\frac{1}{D_1}\pi x)\sin(\frac{1}{D_2}\pi y), \partial_t u(x, y, 0) = u_1(x, y) = 0, \tag{7.2}
$$

$$
\text{with } u(x, y, t) = \sin(\frac{1}{D_1}\pi x)\sin(\frac{1}{D_2}\pi y)\cos(\sqrt{2}\,\pi t) \;\; \text{on} \partial\Omega \times (0, T), \tag{7.3}
$$

where the initial conditions can be written as $u(x, y, t^n) = u_0(x, y)$ and $u(x, y, t^{n-1}) = u(x, y, t^{n+1}) = u(x, y, \Delta t)$.

The analytical solution is given by

$$
u_{\text{ana}}(x, y, t) = \sin(\frac{1}{D_1}\pi x)\sin(\frac{1}{D_2}\pi y)\cos(\sqrt{2}\,\pi t). \tag{7.4}
$$

For the approximation error we choose the L_1-norm.
The L_1-norm is given by

$$err_{L_1} := \sum_{i,j=1,\ldots,m} V_{i,j} \, |u(x_i, y_j, t^n) - u_{\text{ana}}(x_i, y_j, t^n)|, \tag{7.5}$$

where $u(x_i, y_j, t^n)$ is the numerical and $u_{\text{ana}}(x_i, y_j, t^n)$ is the analytical solution, and $V_{i,j} = \Delta x \; \Delta y$.

Our test examples are organized as follows.

1) The non-stiff case: We choose $D_1 = D_2 = 1$ with a rectangle as our model domain $\Omega = [0,1] \times [0,1]$. We discretize with $\Delta x = 1/16$, $\Delta y = 1/16$ and $\Delta t = 1/32$ and choose our parameter η to satisfy $0 \leq \eta \leq 1$. The exemplary function values u_{num} and u_{ana} are taken from the center of our domain.

2) The stiff case: We choose $D_1 = D_2 = 0.01$ with a rectangle as our model domain $\Omega = [0,1] \times [0,1]$. We discretize with $\Delta x = 1/32$, $\Delta y = 1/32$ and $\Delta t = 1/64$ and choose our parameter η to satisfy $0 \leq \eta \leq 1$. The exemplary function values u_{num} and u_{ana} are taken from the point $(0.5, 0.5625)$.

The experiments are done with uncoupled standard discretization methods, i.e. finite differences methods for time and space, and with operator splitting methods, i.e. classical operator-splitting methods and LOD methods.

The non-stiff case can be analyzed in the following tables and figures.

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0014	-0.2663	-0.2697
0.1	0.0030	-0.2663	-0.2738
0.3	0.0063	-0.2663	-0.2820
0.5	0.0096	-0.2663	-0.2901
0.7	0.0128	-0.2663	-0.2981
0.9	0.0160	-0.2663	-0.3060
1.0	0.0176	-0.2663	-0.3100

TABLE 7.1
Numerical results for the finite difference method (see 7.1, Dirichlet boundary).

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0014	-0.2663	-0.2697
0.1	0.0030	-0.2663	-0.2738
0.3	0.0063	-0.2663	-0.2820
0.5	0.0096	-0.2663	-0.2901
0.7	0.0129	-0.2663	-0.2982
0.9	0.0161	-0.2663	-0.3062
1.0	0.0177	-0.2663	-0.3102

TABLE 7.2
Numerical results for the classical operator-splitting method (Dirichlet boundary).

The stiff case can be analyzed in the following tables and figures.

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0014	-0.2663	-0.2697
0.1	0.0031	-0.2663	-0.2739
0.3	0.0065	-0.2663	-0.2824
0.5	0.0099	-0.2663	-0.2907
0.7	0.0132	-0.2663	-0.2990
0.9	0.0165	-0.2663	-0.3073
1.0	0.0182	-0.2663	-0.3114

TABLE 7.3
Numerical results for the LOD method (Dirichlet boundary).

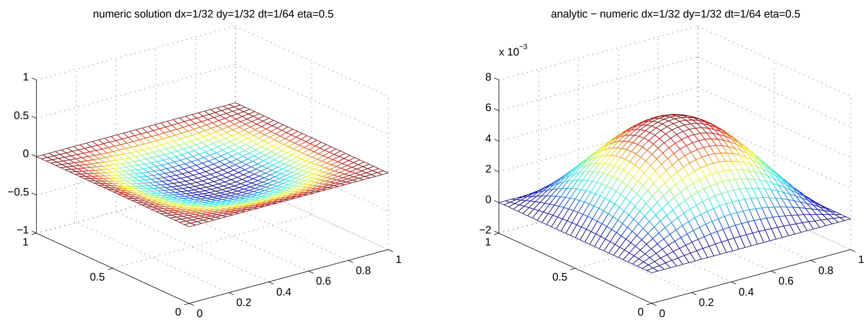

FIG. 7.1. *Numerical resolution for the wave equation: numerical approximation (left) and error functions (right) for the Dirichlet boundary condition ($\Delta x = \Delta y = 1/32$, $\Delta t = 1/64$, $D_1 = 1$, $D_2 = 1$, (classical method).*

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0036	-0.2663	-0.2728
0.1	0.0037	-0.2663	-0.2736
0.3	0.0048	-0.2663	-0.2740
0.5	0.0067	-0.2663	-0.2737
0.7	0.0088	-0.2663	-0.2738
0.9	0.0111	-0.2663	-0.2744
1.0	0.0123	-0.2663	-0.2749

TABLE 7.4
Numerical results for the finite difference method (see 7.1*/ Neumann boundary).*

REMARK 7.1. *In the experiments we compare the non-splitting with the splitting methods. We obtain nearly the same results and see improved results for the LOD method, which is for $\eta = 1/12$ a 4th-order method.*

In the next test example we study the Neumann boundary conditions.

7.2. Test example 2: problem with analytical solution and Neumann boundary condition. In this example we modify our boundary conditions with respect to the Neumann boundary.

We deal with our two-dimensional example where we can derive an analytical

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0036	-0.2663	-0.2728
0.1	0.0037	-0.2663	-0.2736
0.3	0.0048	-0.2663	-0.2740
0.5	0.0067	-0.2663	-0.2737
0.7	0.0089	-0.2663	-0.2738
0.9	0.0112	-0.2663	-0.2745
1.0	0.0123	-0.2663	-0.2750

TABLE 7.5

Numerical results for the classical operator-splitting (see 7.1/ Neumann boundary).

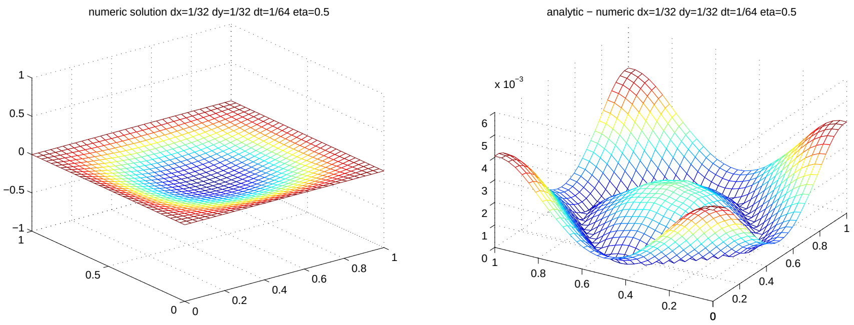

FIG. 7.2. *Numerical resolution for the wave equation: numerical approximation (left) and error functions (right) for the Neumann boundary condition ($\Delta x = \Delta y = 1/32$, $\Delta t = 1/64$, $D_1 = 1$, $D_2 = 1$, (classical method).*

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0335	0.0830	0.2460
0.1	0.0339	0.0840	0.2460
0.3	0.0347	0.0859	0.2460
0.5	0.0354	0.0878	0.2460
0.7	0.0362	0.0896	0.2460
0.9	0.0369	0.0915	0.2460
1.0	0.0373	0.0924	0.2460

TABLE 7.6

Numerical results for the finite difference method for the stiff case with Dirichlet boundary ($\Delta x = \Delta y = 1/32$, $\Delta t = 1/64$).

solution.

$$\partial_{tt} u = D_1^2 \partial_{xx} u + D_2^2 \partial_{yy} u, \tag{7.6}$$

$$u(x, y, 0) = u_0(x, y) = \sin(\frac{1}{D_1}\pi x)\sin(\frac{1}{D_2}\pi y)\ , \partial_t u(x, y, 0) = u_1(x, y) = 0, \tag{7.7}$$

$$\text{with } \frac{\partial u(x, y, t)}{\partial n} = \frac{\partial u_{\text{ana}}(x, y, t)}{\partial n} = 0, \text{ on} \partial\Omega \times (0, T), \tag{7.8}$$

where $\Omega = [0, 1] \times [0, 1]$. $D_1 = 1$, $D_2 = 0.5$ and the initial conditions can be written as $u(x, y, t^n) = u_0(x, y)$ and $u(x, y, t^{n-1}) = u(x, y, t^{n+1}) = u(x, y, \Delta t)$.

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0335	0.2460	0.3227
0.1	0.0339	0.2460	0.3236
0.3	0.0347	0.2460	0.3253
0.5	0.0354	0.2460	0.3271
0.7	0.0362	0.2460	0.3288
0.9	0.0369	0.2460	0.3305
1.0	0.0373	0.2460	0.3314

TABLE 7.7

Numerical results for the classical operator-splitting for the stiff case with Dirichlet boundary ($\Delta x = \Delta y = 1/32,\ \Delta t = 1/64$).

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0335	0.2460	0.3227
0.1	0.0341	0.2460	0.3241
0.3	0.0353	0.2460	0.3268
0.5	0.0365	0.2460	0.3295
0.7	0.0377	0.2460	0.3322
0.9	0.0388	0.2460	0.3349
1.0	0.0394	0.2460	0.3362

TABLE 7.8

Numerical results for the LOD method for the stiff case with Dirichlet boundary ($\Delta x = \Delta y = 1/32,\ \Delta t = 1/64$).

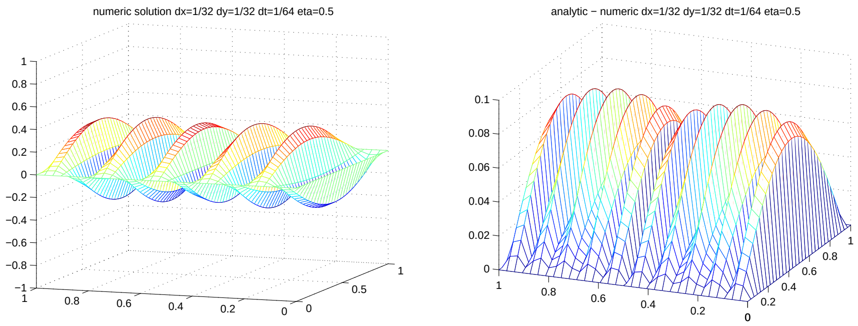

FIG. 7.3. *Numerical approximation and error function for the Dirichlet boundary for the stiff case ($\Delta x = \Delta y = 1/32,\ \Delta t = 1/64,\ D_1 = 1,\ D_2 = 0.01$).*

The analytical solution is given as

$$c_{ana}(x, y, t) = \sin(\frac{1}{D_1}\pi x)\sin(\frac{1}{D_2}\pi y)\cos(\sqrt{2}\,\pi t). \tag{7.9}$$

We have the same discretization methods as in the first test example.

The underlying numerical results for the Neumann boundary conditions are given in Tables 7.9–7.10 and Figure 7.4.

REMARK 7.2. *In the experiments we obtained the same accuracy as for the Dirich-*

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0036	-0.2663	-0.2728
0.1	0.0037	-0.2663	-0.2736
0.3	0.0048	-0.2663	-0.2740
0.5	0.0067	-0.2663	-0.2737
0.7	0.0088	-0.2663	-0.2738
0.9	0.0111	-0.2663	-0.2744
1.0	0.0123	-0.2663	-0.2749

TABLE 7.9
Numerical results for the finite difference method (see 7.1*/ Neumann boundary).*

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0036	-0.2663	-0.2728
0.1	0.0037	-0.2663	-0.2736
0.3	0.0048	-0.2663	-0.2740
0.5	0.0067	-0.2663	-0.2737
0.7	0.0089	-0.2663	-0.2738
0.9	0.0112	-0.2663	-0.2745
1.0	0.0123	-0.2663	-0.2750

TABLE 7.10
Numerical results for the classical operator-splitting (see 7.1*/ Neumann boundary).*

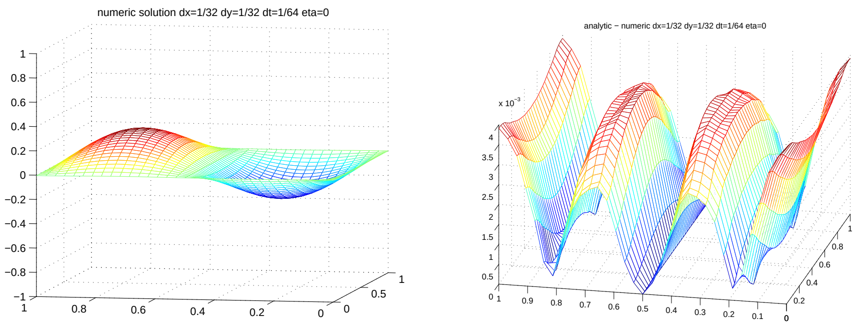

FIG. 7.4. *Numerical resolution for the wave equation: numerical approximation (left) and error functions (right) for the Neumann boundary condition ($\Delta x = \Delta y = 1/32$, $\Delta t = 1/64$, $D_1 = 1$, $D_2 = 1$, (classical method).*

let boundary conditions. More accurate results were gained by the LOD method with small η. We also obtained stable results in our computations.

7.3. Spatially dependent test example. In this experiment we apply our method to the spatially dependent problem, given by

$$\partial_{tt} u = D_1(x,y)\partial_{xx} u + D_2(x,y)\partial_{yy} u, \tag{7.10}$$

$$\text{with } u(x,y,t^n) = u_0, \; \partial_t u(x,y,t^n) = u_1, \tag{7.11}$$

$$\text{with } u(x,y,t) = u_2, \text{ on } \partial\Omega \times (0,T) \tag{7.12}$$

where $D_1(x,y) = 0.1x + 0.01y + 0.01$, $D_2(x,y) = 0.01x + 0.1y + 0.1$.

In this case we are not able to derive an analytical solution. Thus, for comparing the numerical results, we have to compute a reference solution. This can be done with the finite difference scheme with fine temporal and spatial steps.

Concerning the choice of the time steps it is important to consider the CFL conditon, that is now based on the spatial coefficients.

REMARK 7.3. *We have assumed the following CFL condition.*

$$\Delta t < 0.5 \min(\Delta x, \Delta y) / \max_{x,y \in \Omega} (D_1(x,y), D_2(x,y)). \tag{7.13}$$

For the test example we define our model domain as a rectangle $\Omega = [0,1] \times [0,1]$.

The reference solution is obtained by executing the finite difference method and setting $\Delta x = 1/256, \Delta y = 1/256$ as space steps and $\Delta t = 1/256 < 0.390625$ as time step.

The model domain is given by a rectangle with $\Delta x = 1/16$ and $\Delta y = 1/32$. The time steps are given by $\Delta t = 1/16$ and $0 \leq \eta \leq 1$.

The numerical results are given in the following tables and figures.

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0032	-0.7251	-0.7154
0.1	0.0034	-0.7251	-0.7149
0.3	0.0037	-0.7251	-0.7139
0.5	0.0040	-0.7251	-0.7129
0.7	0.0044	-0.7251	-0.7120
0.9	0.0047	-0.7251	-0.7110
1.0	0.0049	-0.7251	-0.7105

TABLE 7.11

Numerical results for the finite difference method with spatially dependent parameters and Dirichlet boundary (error to the reference solution).

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0032	-0.7251	-0.7154
0.1	0.0034	-0.7251	-0.7149
0.3	0.0037	-0.7251	-0.7139
0.5	0.0040	-0.7251	-0.7129
0.7	0.0044	-0.7251	-0.7120
0.9	0.0047	-0.7251	-0.7110
1.0	0.0049	-0.7251	-0.7105

TABLE 7.12

Numerical results for the classical operator-splitting method with spatially dependent parameters and Dirichlet boundary (error to the reference solution).

REMARK 7.4. *In the experiments we analyzed the classical operator-splitting and the LOD method and showed that the LOD method yields yet more accurate values.*

8. Conclusions and discussions. We have presented different time splitting methods for the spatially dependent case of the wave equation. The contributions

η	err_{L1}	u_{ana}	u_{num}
0.00	0.0032	-0.7251	-0.7154
0.1	0.7809e-003	-0.7251	-0.7226
0.122	0.6793e-003	-0.7251	-0.7242
0.3	0.0047	-0.7251	-0.7369
0.5	0.0100	-0.7251	-0.7512
0.7	0.0152	-0.7251	-0.7655
0.9	0.0205	-0.7251	-0.7798
1.0	0.0231	-0.7251	-0.7870

TABLE 7.13

Numerical results for the LOD method with spatially dependent parameters and Dirichlet boundary (error to the reference solution).

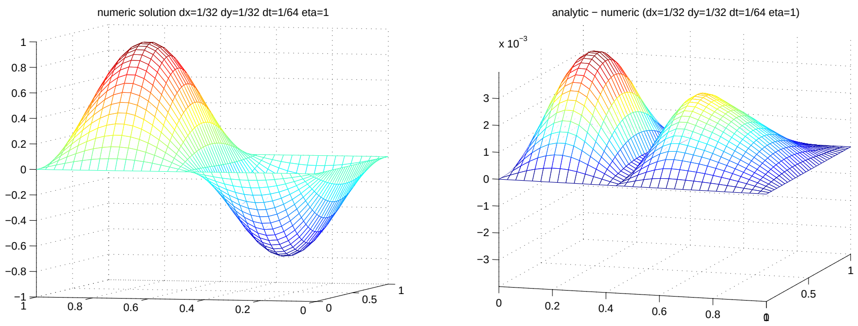

FIG. 7.5. *Dirichlet boundary condition: numerical solution and error function for the spatially dependent test example.*

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0180	-0.7484	-0.7545
0.1	0.0182	-0.7484	-0.7532
0.3	0.0185	-0.7484	-0.7504
0.5	0.0190	-0.7484	-0.7477
0.7	0.0194	-0.7484	-0.7451
0.9	0.0199	-0.7484	-0.7425
1.0	0.0201	-0.7484	-0.7412

TABLE 7.14

Numerical results for the classical operator-splitting method with spatially dependent parameters and Neumann boundary (error to the reference solution).

of this article concern the boundary splitting and the stiff operator treatment. For the boundary splitting method we have discussed the theoretical background and the experiments show the stability of these splitting methods also for the stiff case. We have presented stable results even for the spatially dependent wave equation. The benefit of the splitting methods is due to the different scales and therefore the computational process in decoupling the stiff and non-stiff operators into different equations is accelerated. The LOD method as a 4th-oder method has the advantage of higher accuracy and can be used for such decoupling regards. In a next work we

η	err_{L1}	u_{ana}	u_{num}
0.0	0.0180	-0.7484	-0.7545
0.1	0.0182	-0.7484	-0.7532
0.3	0.0185	-0.7484	-0.7504
0.5	0.0190	-0.7484	-0.7477
0.7	0.0194	-0.7484	-0.7451
0.9	0.0199	-0.7484	-0.7425
1.0	0.0201	-0.7484	-0.7412

TABLE 7.15

Numerical results for the finite difference method with spatially dependent parameters and Neumann boundary (error to the reference solution).

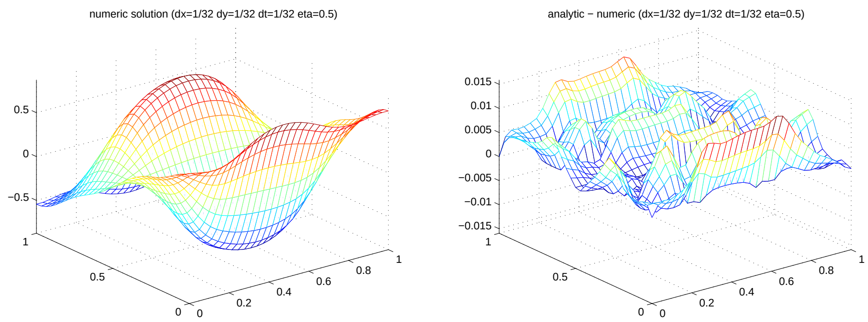

FIG. 7.6. *Neumann boundary condition: numerical solution and error function for the spatially dependent test example.*

discuss the algorithms based on the eigenmodes of the operators for more flexible decoupling problems.

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